J. Fluid Mech. (1973), *vol.* 59, *part* 3, *pp.* 451–479
Printed in Great Britain

Aerodynamic noise emission from turbulent shear layers

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(Received 8 August 1972)

The Phillips (1960) convected wave equation is employed in this paper to study aerodynamic noise emission processes in subsonic and supersonic shear layers. The wave equation in three spatial dimensions is first reduced to an ordinary differential equation by Fourier transformation, then solved via the WKBJ method. Three typical solutions are required for discussions in this paper. The current results are different from the classical conclusions. The effects of refraction, convection, Mach-number dependence and temperature dependence of turbulent noise emission are analysed in the light of solutions to the Phillips equation. Owing to the inherent restrictions of the WKBJ transformation, the results of the present paper should be applied to wave radiation from shear layers whose thickness is no less than approximately one quarter of a wavelength. Such a condition is satisfied for turbulent round jets with an exit velocity greater than 0·6 times the ambient speed of sound.

1. Introduction

This analysis is based on the convected wave equation first introduced by Phillips in 1960. It is intended here to study the noise emission and propagation properties in shear layers with known turbulence structures. An effort has been made to keep the analysis in direct parallel with the classical theory of aerodynamic noise. However, no direct comparison with the Lighthill theory is made because some basic assumptions are different, and the implication of such differences has not been determined. The analytical results indicate several new aspects of noise radiation mechanisms which are not available in the classical results. Since the analysis and physical interpretation of this study are rather involved, the assumptions made at various points throughout this paper are summarized as follows.

The convected wave equation itself is derived through the basic principles of fluid mechanics, and it is a natural extension of the Lighthill equation of aerodynamic noise. The linearized version of the general equation has the form of a simple wave equation in Lagrangian co-ordinates. The right-hand side of this equation contains four terms: a turbulent quadrupole, shear flow and turbulence interaction, entropy fluctuation and viscous effect. If the flow domain is free of shocks, the acoustic pressure fluctuation can be assumed to be decoupled from the entropy fluctuations. It is tacitly assumed in the present analysis that all terms on the right-hand side of the wave equation are known quantities and the contributions of individual terms can be considered as independent of each other. Justification for such an assumption is a difficult fundamental question in the theories of aerodynamic noise.

The configuration of the flow field brings further restrictive assumptions to the convected wave equation such that it becomes mathematically manageable. The shear-layer profile and the turbulence structure are assumed to be homogeneous in time and in the two Cartesian co-ordinates in the plane of the shear layer. The mean flow velocity and temperature profiles are functions of the transverse spatial co-ordinate only. Under such conditions, the three-dimensional wave equation can be reduced to an ordinary differential equation by performing Fourier transformation in the three co-ordinates for which the flow properties are homogeneous. The unknown wave function now depends on only one independent variable, a spatial co-ordinate, together with two wavenumber components and frequency as parameters.

The ordinary differential equation thus obtained is a canonical Sturm–Liouville equation. In essence, this equation can be visualized as a simple wave equation with a variable wavenumber which takes both real and imaginary values. The point at which the wavenumber passes from one domain to another is called a transition point. In the present analysis, this one-dimensional wave equation is first transformed to a standard form by using a WKBJ transformation, and then solved by using the Green function and integral equation technique. Several transformations are employed as required for different numbers of transition points at different positions along the wave propagation path.

A crucial point to be discussed here is the inherent limitations introduced by the WKBJ method. According to Morse & Feshbach (1953), the WKBJ method is applicable only when the change in wavenumber in one wavelength is sufficiently small. In the present analysis, it means that the shear-layer thickness should be at least of the order of one wavelength. However, this comment applies mainly to cases where only the first approximation for the solution is taken. The accuracy of such a solution is therefore subject to an asymptotic condition where the solution is exact only if the wavenumber approaches infinity. In the present analysis, the WKBJ method is used in the transformation of the differential equation, while the equation is solved through the Green function and integral equation technique. Hence, for any given wavenumber the accuracy of the solution can be improved by taking higher terms of the iterations. For practical computations, the convergence rate should be sufficiently rapid as long as the shear-layer thickness is greater than one radian of the acoustic wavenumber, i.e. $kL > 1$, where k is the wavenumber and L is the shear-layer half-thickness. In the case of jet noise, the value of kL is directly proportional to the Strouhal number. It can be shown that the above restriction on wavenumber is satisfied for all noise radiation at frequencies higher than the peak of the noise spectrum if the jet exit velocity is greater than 0·6 times the ambient speed of sound. Hence, the above condition does not impose any significant limitation on the application of the present analysis in dealing with noise radiation from high-speed turbulent shear flows.

By solving the reduced wave equation through the WKBJ method, a solution for the wave function is obtained in mixed variables: one spatial co-ordinate, two wavenumber components and frequency. In the far field, the pressure function approaches asymptotically a function harmonic in the transverse spatial co-ordinate. If the deviation from the harmonic function in the near field is ignored, the wave function can be transformed by Fourier analysis into a pure harmonic representation in wavenumber–frequency co-ordinates. However, neither of the above two representations is convenient for practical applications because it is much more important to know the noise spectrum and intensity at a given point in space. It is therefore necessary to transform the results back through an inverse Fourier transformation into physical co-ordinates. This has been accomplished in closed form. One assumption has been made here to simplify the mathematics: the shear layer is assumed to be symmetric in the transverse spatial co-ordinate. The noise radiation above and below the shear layer is the same at all times.

The key to this part of the analysis is that the process is actually a matching of solutions. First, the pressure field is computed through the Phillips wave equation. Second, it is assumed that in the far field the solution is matched by a pressure field which is governed by a simple wave equation throughout the entire space. Hence, the only differences between the two solutions are in the near field. As far as wave propagation in the far field is concerned, it really does not matter what has happened in the near field. The solution in the far field as obtained through the inverse transformation is indeed the correct solution to the Phillips equation. A by-product of this analysis is that an equivalent source function can be defined. After matching the solution of the Phillips equation in the far field with a solution which is governed by the simple wave equation, a source function in the framework of the simple wave equation can be identified among the formulae. Such an equivalent source function in the near field would provide the best indication of what effect the shear flow has conferred upon the radiation efficiency of the turbulent sound sources.

At this point, the main part of the analysis is complete. Since the non-homogeneous nature of the wave equation in the transverse direction precludes a clear description of the wave propagation process in common terminology of ray acoustics, it is necessary to make some further simplifying assumptions for the sake of interpretation of results. These assumptions include taking the high frequency limit, the definition of the turbulence structure as a Gaussian distribution, order-of-magnitude estimates of integrals, and others. These miscellaneous items will be discussed individually as they are needed later in this paper.

In a review by Laufer, Ffowcs Williams & Childress (1964), it was noted that the original Phillips solution contained a factor $\{M^2-k^2/Q^2\}^{-\frac{1}{4}}Q^{-\frac{1}{2}}$. If the angle ϕ between the x_1 axis and the projection of the wavenumber vector on the plane of the shear layer reaches a critical value such that $1-M_c\cos\phi = 0$, M^2-k^2/Q^2 vanishes. Therefore, the Phillips solution becomes singular. Furthermore, the spectrum of the radiated sound diverges as the factor Q approaches zero. It was on this basis that Laufer, Ffowcs Williams & Childress had strong reservations as to the validity of the Phillips solution.

Both of these singularities can be resolved as they are mathematical in nature.

The occurrence of the former singularity is a result of an assumption made in the analysis of the original Phillips solution. In Phillips (1960), the Mach number is assumed to be very large such that the distance between the pair of transition points is small compared with the thickness of the shear layer. However, the effective convection Mach number $M_c \cos \phi$ equals one at the critical angle. The separation between the transition points then equals the thickness of the shear layers, and the above assumption is violated. This point has been discussed in Pao (1972). In the present analysis, the position of the transition point and the proper form of the WKBJ transformation are determined without any mathematically restrictive assumption. Apart from the restrictions discussed earlier in this section, the solutions are uniformly valid for all wavenumbers at all Mach numbers.

The divergence of the spectrum is a mathematical problem associated with the spectral analysis of either the convected wave equation or the simple wave equation. The factor $(Mq)^{-\frac{1}{2}}$ in the spectral solution to the wave equation is an integrable singularity which appears to have no particular physical significance. In all practical applications, the solution to the wave equation should be written in terms of the space and time co-ordinates instead of the spectral co-ordinates. The former solution can be recovered from the latter by means of an inverse Fourier transformation as given in this paper. One finds that such a solution depends explicitly on $(Mq)^{\frac{1}{2}}$. The singular condition no longer exists.

Among the assumptions in this paper, there are two important differences from the Lighthill theory. In the Lighthill theory, it is assumed that the shear flow dimension is much smaller than the wavelength. In this analysis, the shear flow dimension should be at least of the same order as the wavelength. Second, the Lighthill theory was originally derived for convected turbulence at low Mach numbers while the Phillips theory was originally derived for shear flows with very high Mach numbers. Since the Lighthill theory has been subsequently extended by Ffowcs Williams (1963) to the high-speed regime, the second distinction is actually rather ambiguous. Nevertheless, these differences in the basic assumptions make it rather difficult to compare directly the results of the present analysis with the classical results. In view of such difficulties in comparison, the analysis in the present paper has been kept in direct parallel with the spectral analysis of classical aerodynamic noise theory. By comparing the analysis step by step, one can indeed gain much insight into the difference between these theories.

2. Formulation

By using the momentum equation, the continuity equation and the equation of state for a perfect gas, a convected wave equation can be derived (Phillips, 1960):

$$\frac{D^2}{Dt^2}\log\left(\frac{p}{p_0}\right)-\frac{\partial}{\partial x_i}\left\{c^2\frac{\partial}{\partial x_i}\log\left(\frac{p}{p_0}\right)\right\} = \gamma\frac{\partial u_i}{\partial x_j}\frac{\partial u_j}{\partial x_i}+\gamma\frac{D}{Dt}\left(\frac{1}{c_p}\frac{DS}{Dt}\right) -\gamma\frac{\partial}{\partial x_i}\left\{\frac{1}{\rho}\frac{\partial}{\partial x_j}\mu\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}-\frac{2}{3}\frac{\partial u_k}{\partial x_k}\delta_{ij}\right)\right\}, \quad (1)$$

where p is the pressure, S is the entropy, u_i denote the velocity components, t and x_i denote the time and three-dimensional Cartesian co-ordinates, and c_p, γ and μ are the specific heat at constant pressure, the specific heat ratio and the coefficient of viscosity, respectively. For sound radiation processes in a turbulent flow, both heat conduction and viscosity are likely to be unimportant. Furthermore, if the flow field is shock free, one can probably consider separately the effect of pressure fluctuations and the effect of entropy fluctuations. Under these circumstances, the last two terms of the right-hand side of (1) can be omitted.

In order to construct a solution to the convected wave equation, it is necessary to specify the flow field. A parallel shear flow has been chosen such that it has a characteristic thickness $2L$ and such that the mean flow properties and the turbulent structure are homogeneous in time and the spatial co-ordinates x_1 and x_2 in the plane of the shear layer. The mean flow velocity $\overline{u}_1$ and the local speed of sound a are functions of x_3 only. Although the foregoing assumption should be sufficient, further restrictions can reduce the bulk of the analysis without loss of generality. In Phillips (1960), an antisymmetric flow field is assumed. In the present analysis, the flow is restricted to symmetrical profiles. The fluid far from the shear layer is assumed to be stationary. Equation (1) can be further simplified if small terms are omitted. In a turbulent flow, the fluctuating velocity components are small in comparison with the mean velocity. However, the derivatives of the fluctuating velocity components cannot be assumed small. In the present study, terms depending on small quantities to second or higher orders are omitted. Equation (1) then becomes

$$\left\{\left(\frac{\partial}{\partial t}+\overline{u}_1\frac{\partial}{\partial x_1}\right)^2+\left(\frac{\partial}{\partial t}+\overline{u}_1\frac{\partial}{\partial x_1}\right)u_i'\frac{\partial}{\partial x_i}-\frac{\partial}{\partial x_i}c^2\frac{\partial}{\partial x_i}\right\}\log\left(\frac{p}{p_0}\right) = \gamma\left\{2\frac{\partial \overline{u}_1}{\partial x_2}\frac{\partial u_2'}{\partial x_1}+\frac{\partial u_i'}{\partial x_j}\frac{\partial u_j'}{\partial x_i}\right\}, \quad (2)$$

where u_i' denotes velocity fluctuations with zero mean and $\overline{u}_i$ the mean velocity. Equation (2) is different from the original equation given by Phillips in two respects.

(i) There are two source terms on the right-hand side of this equation. The first term is the shear noise and the second term is the self-noise.

(ii) An additional term $\{(\partial/\partial t+\overline{u}_1\,\partial/\partial x_1)\,u_i'\,\partial/\partial x_i\}\log(p/p_0)$ appears on the left-hand side. It can be regarded as a dispersion term. Both $\partial/\partial t$ and $\overline{u}_1\,\partial/\partial x_1$ of the fluctuating velocity components are large quantities. Fortunately, their combination represents the evolution of the turbulence in the moving frame of reference, which is known to be slow. The value of Du_i'/Dt is of the same order of magnitude as the acoustic radiation. Hence, the effect produced by this term may be compared to the diffraction of sound by sound, which is a second-order effect. After this term has been neglected the resulting equation has the same left-hand side as that given by Phillips.

In the present study, the right-hand side of (2) is assumed known, the same assumption as was made in the Lighthill theory. Such an assumption is made here strictly in the belief that the results of the analysis indeed represent the first-order effects of turbulent noise radiation. Philosophically, the turbulent source function is a part of the dynamical process in the flow, and thus inseparable from the left-hand side of the differential equation. A rational discussion of the limitations of this assumption must be dealt with from a point of view quite different from the practical intentions of the present analysis, and thus lies beyond the scope of this paper.

The shear-layer half-thickness L and the maximum shear-layer velocity U can be chosen as the reference parameters to non-dimensionalize (2). Since the mean flow and turbulence are homogeneous in time and in the plane of the shear layer, a generalized Fourier transform in these co-ordinates is adopted to simplify the non-dimensional convected wave equation. The ordinary differential equation thus obtained can be written as

$$\frac{d^2}{dy^2}\Phi(y,k_1,k_2,\omega)+M^2q^2\Phi(y,k_1,k_2,\omega) = -\frac{M^2}{A^2(y)}\Gamma(y,k_1,k_2,\omega), \tag{3}$$

with

$$\left.\begin{aligned} q^2 &= \left\{\frac{M\omega+k_1M_c(y)}{MA(y)}\right\}^2-\frac{k_1^2+k_2^2}{M^2},\\ v_i = \frac{u_i'}{U},\quad y_i &= \frac{x_i}{L},\quad \tau=\frac{tU}{L},\quad M=\frac{U}{c_0},\quad M_c(y)=\frac{\bar{u}_1(y)}{c_0},\end{aligned}\right\} \tag{4}$$

where y is an abbreviation for y_3: M is the reference Mach number; k_i denotes the wavenumber components and ω stands for frequency. The functions A, Φ and Γ are defined by

$$A(y) = c(y)/c_0,$$

$$\zeta(\mathbf{y},\tau) = \iiint \Phi(y,k_1,k_2,\omega)\exp i(k_1y_1+k_2y_2+\omega\tau)\,dk_1\,dk_2\,d\omega,$$

$$G(\mathbf{y},\tau) = \iiint \Gamma(y,k_1,k_2,\omega)\exp i(k_1y_1+k_2y_2+\omega\tau)\,dk_1\,dk_2\,d\omega,$$

$$\zeta(y,\tau) = A(y)\log\{p(\mathbf{y},\tau)/p_0\}, \tag{5}$$

$$G(y,\tau) = \gamma A(y)\left\{\frac{\partial v_i}{\partial y_j}\frac{\partial v_j}{\partial y_i}+\frac{2}{M}\frac{\partial M_c}{\partial y_2}\frac{\partial v_2}{\partial y_1}\right\},$$

where $G(\mathbf{y},\tau)$ represents both the self-noise and shear-noise source terms in the convected wave equation. Equation (5) will be the working equation for further analysis in this paper. Very few assumptions, apart from linearization and the choice of the flow model, have been adopted in the derivation of this equation. No significant restriction has been placed so far on the range of applicable Mach numbers of the flow as chosen for the convected wave equation.

The function Mq can be analytically identified with the wavenumber k_3. If the Lighthill wave equation is subjected to Fourier transformations in the y_1, y_2 and τ co-ordinates, an ordinary differential equation can be obtained:

$$\left.\begin{aligned} d^2p(y,k_1,k_2,\omega)/dy^2+k_3^2p(y,k_1,k_2,\omega) &= -M^2\Gamma(y,k_1,k_2,\omega),\\ k_3^2 &= M^2\omega^2-(k_1^2+k_2^2).\end{aligned}\right\} \tag{6}$$

In the far field, the value of Mq approaches a constant. Hence, (3) and (6) are identical if the function Mq and the wavenumber component k_3 are assumed to stand for the same quantity. Owing to the interaction of the shear flow and the wave propagation process, the function Mq becomes a variable in the near field. However, it can still be identified with the transverse wavenumber component in a limited sense.

The function M^2q^2 may vanish at certain points in the shear layer. Such a zero is called a transition point because the value of M^2q^2 changes sign across this point, and the reduced wave equation changes type either from hyperbolic to elliptic or vice versa. Hence, it is necessary to employ the WKBJ method to solve (3). There are three cases that need to be considered. In the first case, there is no transition point along the path between the sound source and the far field. In common practice, it is actually not necessary to use the WKBJ technique to solve (3) in this case. Here it is considered as a zeroth-order WKBJ transformation and is treated in the same manner as the other two cases. In the second case, there is one transition point located between the sound source and the far field. However, there should be no other transition point within approximately two radians of wavelength. Here the differential equation will be hyperbolic in the far field and elliptic near the source. The source oscillation which contributes to nose radiation is hydrodynamic in nature. For this case the WKBJ transformation for well-separated transition points will be employed. In the third case, there are two transition points near the sound source. The distance between these two transition points is less than two radians of wavelength. This case can be analysed by a WKBJ transformation for a transition point where M^2q^2 has a zero of second order.

The occurrence of transition points is mainly a function of the convection Mach number and the angle of radiation. The situation can be considered from two different points of view. If a small sound source volume is given in the shear layer, where the M_c can be regarded as constant, the number and position of transition points are determined by the angle of radiation. In order to describe completely the wave radiation in all directions, all of the above conditions for (3) will be encountered. From another point of view, one can consider wave radiation in a given direction with source contributions from all layers of the shear flow. In this case, the transition point will be fixed at a point of the shear layer where M_c is predetermined by the angle of radiation. Although the former point of view is the most practical for applications, the latter view is represented by the formulation of (3).

3. Solution to the convected wave equation

Through the Fourier analysis indicated in the previous section, the pressure field and the sound source function have been broken up into plane-wave elements. Since the Fourier transformation is performed in the stationary frame of reference, the frequency ω represents the frequency of a plane-wave element as measured at a fixed point in the far field. The projection of the wavenumber vector on the plane of the shear layer is represented by k_1 and k_2. These components remain constant throughout the entire space including the shear-layer flow domain if a plane-wave element has been chosen in the far field. However, wave propagation is non-uniform in the direction normal to the plane of the shear layer and is governed by (3).

In the far field, a plane-wave element is determined by its frequency and wavenumber vector. In the present case, the frequency and two components of the wavenumber vector are known. Since the magnitudes of the wavenumber and frequency are related through the speed of sound, k_3 is also known. From the above indications, it is clear now that the role of (3) is to determine how a chosen plane-wave element in the far field is related to the turbulent sound sources in the near field, and how sound is propagated from the source through the shear layer into the far-field. The solutions to (3) given below for various cases will confirm the above-mentioned role of the reduced wave equation.

By using a WKBJ transformation, the one-dimensional wave equation can be transformed to the following standard form:

$$d^2\eta/d\xi^2 + \xi^n\eta = g_n(\xi)\,\eta + h_n(\xi) \quad (n = 0, 1, 2), \tag{7}$$

with

$$\left.\begin{aligned} g_n(\xi) &= \tfrac{1}{2}\psi'^{-2}\{\psi'''/\psi' - \tfrac{3}{2}(\psi''/\psi')^2\}, \\ h_n(\xi) &= -M^2\Gamma(\xi, k_1, k_2, \omega)/\psi'^{\frac{3}{2}}A^2, \end{aligned}\right\} \tag{8}$$

where the transformations of the independent and dependent variables are defined by

$$\begin{gathered} \eta(\xi) = \psi'(y)\,\Phi(y), \\ \xi = \psi(y) = \left\{\frac{n+2}{2}\int_{y_0}^{y} Mq(y)\,dy\right\}^{2/(n+2)}, \\ \psi'(y) = d\xi/dy = q(y)\,\xi^{-\frac{1}{2}n}. \end{gathered} \tag{9}$$

The value of n denotes the order of zero at the transition point, and the lower integral limit y_0 indicates the position of the transition point. For $n = 0$, there is actually no transition point throughout the path of integration. Therefore, y_0 can be conveniently chosen as $y_0 = 0$. The function $g_n(\xi)$ is a residue function arising from the WKBJ transformation. In the far field, $g_n(\xi)$ approaches zero as ξ^{-2}. In the neighbourhood of the transition point, the limiting values of ψ' and g_n remain finite as ξ approaches zero. In the remainder of this paper, the solutions will be identified by $S0$, $S1$ and $S2$ for cases with $n = 0$, 1 and 2, respectively.

Since the Green function, integral equation technique for solving (3) is essentially the same as that given in Phillips (1960), details of the derivations will not be repeated here. In Pao (1971), the solution to (3) has been given as

$$\eta(\xi) = H(\xi) + \int_0^\xi R(\xi, s)\,\{g_n(s)\,H(s) + h_n(s)\}\,ds, \tag{10}$$

where $H(\xi)$ is an arbitrary solution which satisfies the homogeneous wave equation and $R(\xi, s)$ is the resolvent kernel which is conjugate to the kernel $K(\xi, s)$ of the integral equation. As a first approximation, the resolvent kernel

$$R(\xi, s) = K(\xi, s).$$

However, the solution can actually be given in a simpler form:

$$\eta(\xi) = H(\xi) + \int_0^\xi R(\xi, s)\,h(s)\,ds. \tag{11}$$

The difference lies in the starting-point of the iteration. In (10), the zeroth-order solution is assumed to be $H(\xi)$, while in (11) the zeroth-order solution is assumed to be zero. In the limit, both iterations will lead to the same analytic solution. For each of the three main solutions, the boundary conditions and the analytical nature of the integrals are different. These solutions will be discussed separately. For clarity of interpretation of results, only the first approximation to the resolvent kernel will be employed in the discussions immediately below. The higher iterations will be discussed later in this paper.

The acoustic mode S0

In the $S0$ solution, no transition point is encountered along the wave propagation path. Hence, the value of M^2q^2 is always positive and bounded away from zero. The homogeneous solutions to (3) are the simple harmonic functions of ξ. The kernel of the integration can be written as

$$K(\xi, s) = (2i)^{-1}\{e^{i(\xi-s)} - e^{-i(\xi-s)}\} = \sin(\xi - s), \tag{12}$$

where s is a dummy variable corresponding to ξ. It should be noted also that the origin of the ξ co-ordinate has been defined to be the same as the origin of the y_3 co-ordinate. Since the first approximation to the resolvent kernel is $K(\xi, s)$, the solution can be given as

$$\Phi_0(y, k_1, k_2, \omega) = \{Mq(y)\}^{-\frac{1}{2}}\left\{a\, e^{i\xi} + b\, e^{-i\xi} + \int_0^{\xi} \sin(\xi - s) h_0(s, k_1, k_2, \omega)\, ds\right\}, \tag{13}$$

where a and b are arbitrary constants to be determined by the boundary conditions.

If the wave is assumed to be propagating away from the shear layer at $y = \pm\infty$ these radiation conditions will serve as the required two boundary conditions. In the present discussion, the shear layer is similar to a jet exhaust flow into a stationary ambient medium. The convection velocity vanishes on the y_3 axis at $y_3 = \pm\infty$. The main shear flow will be confined to the neighbourhood of the y_1, y_2 plane.

For the above-mentioned flow condition, it is easy to verify that the outgoing wave at $+\infty$ is represented by $\exp(i\xi)$ and the outgoing wave at $-\infty$ by $\exp(-i\xi)$ The boundary conditions can be written as

$$\left.\begin{aligned} a + \frac{1}{2i}\int_0^{-\infty} e^{-is} h_0(s, k_1, k_2, \omega)\, ds &= 0,\\ b - \frac{1}{2i}\int_0^{\infty} e^{is} h_0(s, k_1, k_2, \omega)\, ds &= 0.\end{aligned}\right\} \tag{14}$$

Hence, the solution to the convected wave equation for the $S0$ mode is

$$\Phi_0(y, k_1, k_2, \omega) = (2i)^{-1}\{Mq(y)\}^{-\frac{1}{2}} \times \left\{e^{i\xi}\int_{-\infty}^{\xi} e^{-is} h(s, k_1, k_2, \omega)\, ds + e^{-i\xi}\int_{\xi}^{\infty} e^{is} h(s, k_1, k_2, \omega)\, ds\right\}. \tag{15}$$

From the structure of this solution, some immediate conclusions can be made. First of all, a local frequency can be defined through a Lagrangian transformation of co-ordinates. The local frequency of pressure fluctuations as measured in a frame of reference which moves with the convection velocity M_c can be written as

$$\omega_0 = \omega + k_1 M_c/M. \tag{16}$$

Once a far-field plane-wave element has been determined, the values of k_1, k_2 and ω will remain constant. The value of Mq, though variable, is directly proportional to the value of ω. In the high frequency limit, there is a straightforward physical interpretation of (15). The value of Mq can be made arbitrarily large as the value of ω increases. Under such conditions, the value of Mq will appear to be constant over many wavelengths in a small interval Δy in the shear layer. If it is further assumed that a correlated turbulence volume lies in Δy, then the integrals in (15) will have the form of a Fourier transformation. These integrals will thus serve to select a harmonic component from the source function and pass it to the far field as a radiated plane-wave element. This source element will have a local frequency ω_0 and a wavenumber vector (k_1, k_2, Mq). However, by definition of Mq (equations (3) and (16)), the following relation holds:

$$\omega_0^2 = (A/M)^2\{k_1^2 + k_2^2 + (Mq)^2\}. \tag{17}$$

Hence the magnitude of the wavenumber vector is related to the local frequency through the local speed of sound. The source element which is responsible for far-field noise radiation is, therefore, an acoustic component of the turbulence structure in the convected frame of reference. Therefore, the $S0$ mode of noise emission can also be called the acoustic mode.

The Doppler effect of frequency shift can be recovered directly from (16). If the angle between the far-field wavenumber vector and the y_1 axis is defined as θ, equation (16) can be written as

$$\omega_0 = \omega(1 - M_c\cos\theta), \quad \text{with} \quad k_1 = k_\infty \cos\theta, \quad \omega = -k_\infty/M, \tag{18}$$

where k_∞ denotes the magnitude of the wavenumber vector in the far field, and the negative sign in the wavenumber–frequency ratio indicates a forward-propagating wave according to the definition of the Fourier transformations. Therefore

$$\omega = \omega_0/(1 - M_c\cos\theta). \tag{19}$$

Since it is known from the above discussion that the local frequency of the source function equals ω_0, the source function and the far-field noise radiation are related by the Doppler shift relation. Equation (19) is derived only from the far-field conditions and the definition of ω_0, its validity does not depend on the details of the shear flow profile. This relation should apply equally well to solutions $S1$ and $S2$.

The S1 solution

In this case one transition point exists between the far field and the source region. The governing differential equation is hyperbolic on the far-field side of the transition point and is elliptic on the other side. In Erdélyi (1956, p. 98) it is shown that the solutions on both sides of the transition point can be made analytically continuous through the use of Airy functions. The matching condition at the transition point has been included in the definition of the Airy functions. For this reason, it is more convenient to define the transformation in a way which is slightly different from (9):

$$\xi(y) = \left\{\tfrac{3}{2}\int_{y_0}^{y} M|q|\,dy\right\}^{\frac{2}{3}}. \tag{20}$$

According to this definition, ξ is real and positive for points on either side of the transition point. The sign convention for the Airy functions requires, however, that the argument of the functions be negative for the hyperbolic branch of the solution and positive for the elliptic branch. In the present analysis, the ξ co-ordinate is pointing at the opposite direction from the y co-ordinate for wave radiation to plus infinity.

The kernel for the integral equation can be given as

$$K(\xi, s) = \begin{cases} \pi\{\mathrm{Bi}\,(-\xi)\,\mathrm{Ai}\,(-s) - \mathrm{Ai}\,(-\xi)\,\mathrm{Bi}\,(-s)\} & \text{for} \quad y > y_0, \\ \pi\{\mathrm{Bi}\,(\xi)\,\mathrm{Ai}\,(s) - \mathrm{Ai}\,(\xi)\,\mathrm{Bi}\,(s)\} & \text{for} \quad y \leqslant y_0. \end{cases} \tag{21}$$

Hence, the first-order approximate solutions to the convected wave equation are

$$\Phi_1(y, k_1, k_2, \omega) = \{Mq(y)\}^{-\frac{1}{2}}\xi^{\frac{1}{4}}\Big\{a\,\mathrm{Ai}\,(-\xi) + b\,\mathrm{Bi}\,(-\xi) + \int_0^{\xi} \pi\,\{\mathrm{Bi}\,(-\xi)\,\mathrm{Ai}\,(-s) - \mathrm{Ai}\,(-\xi)\,\mathrm{Bi}\,(-s)\}\,h_1(s, k_1, k_2, \omega)\,ds\Big\} \quad \text{for} \quad y > y_0 \tag{22}$$

and

$$\Phi(y, k_1, k_2, \omega) = \{Mq(y)\}^{-\frac{1}{2}}\xi^{\frac{1}{4}}\Big\{a\,\mathrm{Ai}\,(\xi) + b\,\mathrm{Bi}\,(\xi) + \int_0^{\xi} \pi\,\{\mathrm{Bi}\,(\xi)\,\mathrm{Ai}\,(s) - \mathrm{Ai}\,(\xi)\,\mathrm{Bi}\,(s)\}\,h_1(s, k_1, k_2, \omega)\,ds\Big\} \quad \text{for} \quad y \leqslant y_0. \tag{23}$$

Two boundary conditions are required for the $S1$ solution: the pressure wave is outgoing as ξ approaches minus infinity, and the pressure fluctuation vanishes as ξ approaches plus infinity. In order to define the boundary conditions, the asymptotic approximations for the Airy functions with large argument are required:

$$\begin{aligned} \mathrm{Ai}\,(-\xi) &\sim \pi^{-\frac{1}{2}}\xi^{-\frac{1}{4}}\sin(\zeta + \tfrac{1}{4}\pi), \quad \mathrm{Bi}\,(-\xi) \sim \pi^{-\frac{1}{2}}\xi^{-\frac{1}{4}}\cos(\zeta + \tfrac{1}{4}\pi), \\ \mathrm{Ai}\,(\xi) &\sim \tfrac{1}{2}\pi^{-\frac{1}{2}}\xi^{-\frac{1}{4}}e^{-\zeta}, \quad \mathrm{Bi}\,(\xi) \sim \pi^{-\frac{1}{2}}\xi^{-\frac{1}{4}}e^{\zeta}, \end{aligned} \tag{24}$$

where $\zeta = \frac{2}{3}\xi^{\frac{3}{2}}$. After these values of the Airy functions have been substituted into (22) and (23), it can be seen that the asymptotic value of the solution is a linear combination of the simple harmonic functions $\exp\{\pm i\int M|q|\,dy\}$ in the far field, and the exponential functions $\exp\{\pm i\int M|q|\,dy\}$ in the near field. Hence, the boundary conditions can be written as

$$\begin{aligned} a + \pi\int_0^{\infty} \mathrm{Bi}\,(-s)\,h_1(s)\,ds - ib - \pi i\int_0^{\infty} \mathrm{Ai}\,(-s)\,h_1(s)\,ds &= 0, \\ b + \int_0^{\infty} \pi\,\mathrm{Ai}\,(s)\,h_1(s)\,ds &= 0, \end{aligned} \tag{25}$$

where $h_1(s)$ is an abbreviation of $h_1(s, k_1, k_2, \omega)$. The solution to the $S1$ case for $y \geqslant y_0$ in the far field can now be written as

$$\Phi_1(y, k_1, k_2, \omega) = \pi\{Mq(y)\}^{-\frac{1}{2}}\xi^{\frac{1}{4}}\Big[\{\mathrm{Bi}\,(-\xi)+i\,\mathrm{Ai}\,(-\xi)\} \times\left\{-\int_0^\infty \mathrm{Ai}\,(s)\,h_1(s)\,ds+\int_0^\xi \mathrm{Ai}\,(-s)\,h_1(s)\,ds\right\} + \mathrm{Ai}\,(-\xi)\int_\xi^\infty \{\mathrm{Bi}\,(-s)+i\,\mathrm{Ai}\,(-s)\}\,h_1(s)\,ds\Big]. \quad (26)$$

Since the present subject is wave radiation into the far field, the solution for the elliptic branch is not shown. However, it can be easily recovered from (23) by substituting into it the values of the constants a and b.

In (26), the last term on the right-hand side vanishes as soon as y is beyond the shear layer. The remainder represents the source contribution of the shear layer along the path of wave propagation. The contributions from the elliptic region and the hyperbolic region are given in separate terms. In the elliptic region, the integral represents an exponentially weighted sum of the source elements located at variable distances from the transition point. The radiation efficiency of a source function decreases rapidly with distance from the transition point. It should be noted that the distance is measured in terms of wavelengths. For the same physical distance along the y axis, the distance would appear to be much greater for high frequency than for low frequency. In this region, the transverse wavenumber of the source element is imaginary, and the ratio of frequency to the real component of the wavenumber is less than the local speed of sound. Hence, the source element can be characterized as hydrodynamic.

In the hyperbolic region, the function $\mathrm{Ai}\,(-s)$ is oscillatory. When the value of s is large, it can be shown that the noise radiation solution for the hyperbolic region of $S1$ approaches the form of the $S0$ solution asymptotically, if the function $\mathrm{Ai}\,(-s)$ is replaced by its asymptotic value as given in (24). The interpretation of this part of the solution is, therefore, the same as for the $S0$ mode. From discussions of the contributions of the source function in the elliptic zone and the hyperbolic zone, an interesting conclusion can be made. For low frequencies, source elements in both regions can radiate noise with comparable efficiency. For high frequencies, only sources in the hyperbolic region and those in the neighbourhood of the transition point are responsible for effective emission of noise into the far field.

For shear flows with very large convection Mach numbers, more than one transition point may appear within the half-thickness of the shear layer. The elliptic zone of the differential equation is now bounded. In the presence of a second transition point, the second boundary condition, which states that the pressure fluctuation vanishes at infinity in the elliptic zone, cannot be established. However, the second transition point can be ignored if the frequency of sound under consideration is sufficiently high because the separation between these transition points will appear to be very large, and their influence on each other will be very small. If these transition points are separated by more than two radians of wavelength, the error in the solution due to accepting the second boundary condition in its present form is less than one per cent.

The S2 solution

If the distance between two consecutive transition points is less than two radians, their presence must be considered together. In the limiting case, the transition points are considered to be a single point with a second-order zero for the function M^2q^2. The WKBJ transformation and the resulting differential equation for the present case is given by (7) with $n = 2$. The homogeneous solutions of the wave equation are related to Bessel functions of order one quarter, as given in Phillips (1960). It is more convenient to represent the homogeneous solutions by a pair of functions Pa and Qa:

$$\left.\begin{aligned} \mathrm{Pa}(\xi) &= \tfrac{1}{2}\xi^{\frac{1}{2}}\{J_{-\frac{1}{4}}(\tfrac{1}{2}\xi^2) - \alpha_1 J_{\frac{1}{4}}(\tfrac{1}{2}\xi^2)\}, \\ \mathrm{Qa}(\xi) &= \tfrac{1}{2}\xi^{\frac{1}{2}}\{J_{-\frac{1}{4}}(\tfrac{1}{2}\xi^2) + i\alpha_1 J_{\frac{1}{4}}(\tfrac{1}{2}\xi^2)\}, \\ \alpha_1 &= 2^{-\frac{1}{2}}(1+i). \end{aligned}\right\} \tag{27}$$

The kernel for the integral equation can now be written as

$$K(\xi, s) = \pi\{\mathrm{Qa}\,(\xi)\,\mathrm{Pa}\,(s) - \mathrm{Pa}\,(\xi)\,\mathrm{Qa}\,(s)\}. \tag{28}$$

It should be noted that the reduced wave equation for $S2$ is hyperbolic on either side of the second-order transition point. In the present study, the origin of the WKBJ transformation is defined along the y axis as follows: it is the upper transition point for the upper branch and the lower transition point for the lower branch. In Phillips (1960), the definition of the origin is somewhat different. Neither definition is exact. In the present case, the derivative of the transformation needs to be defined in the neighbourhood of the transition point such that it remains finite:

$$\lim_{\xi\to 0} \psi'(y) = k_s^{\frac{1}{2}}\left\{\frac{d}{dy}\left(\frac{M_c}{A}\right)\right\}^{\frac{1}{2}} \quad (k_s = (k_1^2 + k_2^2)^{\frac{1}{2}}). \tag{29}$$

On the other hand, the Phillips definition results in an inaccuracy of the transverse wavenumber component in the far field. Since the final solution is given in integral form, an isolated singular point in the definition of ψ' can be removed with affecting the value of the solution. The present definition of the WKBJ variable appears to be acceptable. For both the upper and the lower branches, the solution can be written as

$$\begin{aligned} \Phi_2(y, k_1, k_2, \omega) = \{Mq(y)\}^{-\frac{1}{2}}\xi^{\frac{1}{2}}\Big\{&a\,\mathrm{Pa}\,(\xi) + b\,\mathrm{Qa}\,(\xi) \\ &+ \pi\int_0^{\xi}[\mathrm{Qa}\,(\xi)\,\mathrm{Pa}\,(s) - \mathrm{Pa}\,(\xi)\,\mathrm{Qa}\,(s)]\,h_2(s, k_1, k_2, \omega)\,ds\Big\}. \end{aligned} \tag{30}$$

There is a total of four arbitrary constants: a_1 and b_1 for the upper branch, and a_2 and b_2 for the lower branch. These constants will be determined by four boundary conditions. Two of the boundary conditions require that the pressure wave should be outgoing at $y = \pm\infty$. The other two conditions require that the branch solutions should match at $\xi = 0$. At the second-order transition point the pressure should be in equilibrium, and the wave-induced flow field should be kinematically compatible. For $S2$, the convection velocity has a finite jump across the transition point. Its value is exactly twice the local speed of sound.

The treatment of matching conditions for wave propagation across a shear discontinuity has been discussed in detail by Miles (1957) and Ribner (1957). The same matching conditions are used in the present analysis. The asymptotic values of the functions Pa and Qa for large values of ξ can be written as

$$\left.\begin{aligned}\mathrm{Pa}(\xi) &\simeq (2\pi\xi)^{-\frac{1}{2}}\exp\{-i(\tfrac{1}{2}\xi^2+\tfrac{1}{8}\pi)\}\\ \mathrm{Qa}(\xi) &\simeq (2\pi\xi)^{-\frac{1}{2}}\exp\{i(\tfrac{1}{2}\xi^2+\tfrac{1}{8}\pi)\}\end{aligned}\right\} \quad \text{as} \quad \xi\to\infty. \tag{31}$$

If these expressions are substituted into (30), the solution will be represented by the harmonic functions $\exp\{\pm i\int M|q|\,dy\}$ in the far field. There are two cases which need to be considered.

First, if the shear flow velocity is extremely large, two transition points may appear simultaneously within the half-thickness of the shear layer. Since the shear gradient is large, these points may be very close together. The shear flow on the two sides of these transition points appears to flow in different directions. Its analysis is quite similar to flows with an antisymmetrical profile. Here, the outgoing wave is represented by the Qa(ξ) component for both branches. The radiation conditions can now be written as

$$\left.\begin{aligned}a_1-\pi\int_0^{+\infty}\mathrm{Qa}(s)\,h_2(s,k_1,k_2,\omega)\,ds &= 0,\\ a_2-\pi\int_0^{-\infty}\mathrm{Qa}(s)\,h_2(s,k_1,k_2,\omega)\,ds &= 0.\end{aligned}\right\} \tag{32}$$

According to Ribner (1957), the matching conditions at the origin of the ξ coordinate can be given as

$$\left.\begin{aligned}a_1\,\mathrm{Pa}(0)+b_1\,\mathrm{Qa}(0) &= a_2\,\mathrm{Pa}(0)+b_2\,\mathrm{Qa}(0),\\ a_1\,\mathrm{Pa}'(0)+b_1\,\mathrm{Qa}'(0) &= -a_2\,\mathrm{Pa}'(0)-b_2\,\mathrm{Qa}'(0),\\ \mathrm{Qa}(0) = \mathrm{Pa}(0), \quad \mathrm{Qa}'(0) &= -i\,\mathrm{Pa}'(0),\end{aligned}\right\} \tag{33}$$

where Pa′ and Qa′ indicate first derivatives with respect to ξ. It follows that

$$\left.\begin{aligned}b_1 &= \tfrac{1}{2}(1-i)\,a_2-\tfrac{1}{2}(1+i)\,a_1,\\ b_2 &= \tfrac{1}{2}(1-i)\,a_1-\tfrac{1}{2}(1+i)\,a_2.\end{aligned}\right\} \tag{34}$$

Hence, the solution to the wave equation can be written as

$$\Phi_2(y,k_1,k_2,\omega) = \pi\{Mq(y)\}^{-\frac{1}{2}}\xi^{\frac{1}{2}}\Big[\mathrm{Pa}(\xi)\int_\xi^{+\infty}\mathrm{Qa}(s)\,h_2(s)\,ds+\mathrm{Qa}(\xi)$$
$$\times\Big\{\int_0^\xi\mathrm{Pa}(s)\,h_2(s)\,ds+\tfrac{1}{2}(1-i)\int_0^{-\infty}\mathrm{Qa}(s)\,h_2(s)\,ds-\tfrac{1}{2}(1+i)\int_0^{+\infty}\mathrm{Qa}(s)\,h_2(s)\,ds\Big\}\Big]$$

for the upper branch, and (35)

$$\Phi_2(y,k_1,k_2,\omega) = \pi\{Mq(y)\}^{-\frac{1}{2}}\xi^{\frac{1}{2}}\Big[\mathrm{Pa}(\xi)\int_\xi^{-\infty}\mathrm{Qa}(s)\,h_2(s)\,ds+\mathrm{Qa}(\xi)$$
$$\times\Big\{\int_0^\xi\mathrm{Pa}(s)\,h_2(s)\,ds+\tfrac{1}{2}(1-i)\int_0^{+\infty}\mathrm{Qa}(s)\,h_2(s)\,ds-\tfrac{1}{2}(1+i)\int_0^{-\infty}\mathrm{Qa}(s)\,h_2(s)\,ds\Big\}\Big] \tag{36}$$

for the lower branch. In the above equations, the radiated wave is represented by the Qa(ξ) term. Its coefficient is composed of three parts: direct outward radiation from the sources, inward radiation as reflected by the transition zone, and the contribution from sources on the other side of the transition points.

In the second case, the velocity of the shear flow is not necessarily high. Since the velocity profile is symmetric, the transition points come always in pairs. When the frequency of noise radiation is sufficiently low, these transition points have to be considered together. The boundary conditions are different: the solution approaches Qa (ξ) at $y = +\infty$ and Pa (ξ) at $y = -\infty$. The WKBJ transformation near the transition point is also different. Since the velocity gradient vanishes at the transition point, the limiting value of ψ' at $\xi = 0$ is given by

$$\psi'(y_0) = k_s^{\frac{1}{2}}\left\{\frac{1}{2}\frac{d^2}{dy^2}\left(\frac{M_c}{A}\right)\right\}^{\frac{1}{4}}, \tag{37}$$

where the second derivative of M_c/A indicates the mean curvature of the shear flow profile at $\xi = 0$. The matching condition near the origin of the ξ co-ordinate now reads

$$\left.\begin{aligned} a_1 \operatorname{Pa}(0) + b_1 \operatorname{Qa}(0) &= a_2 \operatorname{Pa}(0) + b_2 \operatorname{Qa}(0), \\ a_1 \operatorname{Pa}'(0) + b_1 \operatorname{Qa}'(0) &= a_2 \operatorname{Pa}'(0) + b_2 \operatorname{Qa}'(0), \end{aligned}\right\} \tag{38}$$

which is actually equivalent to stating that both the pressure and the pressure gradient be continuous across the transition point. The solution for the present case can now be written as

$$\Phi_2(y, k_1, k_2, \omega) = \pi\{Mq(y)\}^{-\frac{1}{2}}\xi^{\frac{1}{2}}\left\{\operatorname{Pa}(\xi)\int_{\xi}^{+\infty}\operatorname{Qa}(s)\,h_2(s)\,ds + \operatorname{Qa}(\xi) \times \int_{-\infty}^{\xi}\operatorname{Pa}(s)\,h_2(s)\,ds\right\}. \tag{39}$$

Equations (35), (36) and (39) are analytically similar. In further discussions in this paper, these cases will not be cited separately.

Since the wavenumber is assumed to be small, the value of ξ will be small throughout the shear layer. The source function is concentrated near the origin as far as the ξ co-ordinate is concerned. Furthermore, the interval between the original transition points along the y axis is reduced to zero in the transformation. Hence, it is more convenient to calculate the integrals in terms of the y co-ordinate. In this case, all the sources are located near the origin of the ξ co-ordinate. The expression for the source integral can be significantly simplified. For example, the coefficient of the Qa (ξ) term in (36) can be written as

$$\int_0^{\xi}\operatorname{Pa}(s)\,h_2(s)\,ds + \tfrac{1}{2}(1-i)\int_0^{\xi}\operatorname{Qa}(s)\,h_2(s)\,ds - \tfrac{1}{2}(1+i)\int_0^{+\infty}\operatorname{Qa}(s)\,h_2(s)\,ds = \tfrac{1}{2}(1-i)\int_{-1}^{1}\operatorname{Pa}(0)\frac{M^2\Gamma(y, k_1, k_2, \omega)}{\psi'^{\frac{1}{2}}A^2}\,dy, \tag{40}$$

where Pa (0) = 0·5770338. The coefficients for the Qa (ξ) term in (37) and (30) can also be reduced to the same form, where the limits of integration in the y co-ordinate are the boundaries of the shear layer. In the interval between the original transition points, the source function can be identified as a hydrodynamic pressure fluctuation. The source in this interval is in fact the most important part for $S2$. In the above equation, the source integral is simply a sum of all the contributions throughout the shear layer without weighting factors. In common turbulence structures, the hydrodynamic components contain most of the kinetic energy. Hence, the contribution to noise radiation comes mainly from source functions in the elliptical region of the $S2$ wave equation.

Although the $S2$ analysis is based on the assumption of low-frequency noise radiation, the validity of the results here is limited. In the WKBJ transformation, the residue function $g(\xi)$ is proportional to k^{-2} within the shear layer. It becomes very large if k is much smaller than one. Although the iteration of the integral equation will eventually converge, the convergence rate will be very poor. For all practical purposes, the present solution should be applied only to cases where $k \sim 1$.

The inverse Fourier transformation

In most practical studies of noise radiation from turbulence, it is necessary to know the sound intensity and spectrum at given points in space. The solutions obtained so far in the present paper are represented mainly in terms of the wavenumber–frequency co-ordinates. Hence, it is the purpose of the analysis in this section to recover the spatial resolution of the $S0$, $S1$ and $S2$ wave functions. The spatial representation of the solution is advantageous from yet another point of view. It has been noted before in Laufer *et al.* (1964) that the spectral solution diverges at small values of Mq. Evidently, a factor of $(Mq)^{-\frac{1}{2}}$ is contained in all solutions for $S0$, $S1$ and $S2$ as given by (15), (26), (36), (37) and (39). This weak singularity has stirred up serious concern about the validity of the solutions. Such a singularity will not appear in the spatial representation of the solution to the wave equation.

The solutions in all three cases approach asymptotically the harmonic functions $\exp(\pm i \int Mq(y)\,dy)$ in the far field. It is possible to represent $\Phi_n(y, k_1, k_2, \omega)$ entirely in terms of the wavenumber–frequency co-ordinates. In order to keep the mathematical expressions from being too complicated, the shear flow profile and its associated turbulence structure are assumed to be symmetric with respect to the y co-ordinate. As a consequence, the radiated wave field can also be regarded as symmetric with respect to the y co-ordinate. With this assumption, the Fourier transformation of the wave function with respect to y will be the same as a cosine transform, and it can be written as (Erdélyi 1954)

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \Phi_n(y, k_1, k_2, \omega)\, e^{-ik_3 y}\, dy = \{Mq_\infty\}^{\frac{1}{2}} \frac{1}{k_3^2 - (Mq_\infty)^2}\, \Phi_n(k_1, k_2, Mq_\infty, \omega)$$

$$= \{Mq_\infty\}^{\frac{1}{2}} \frac{1}{k_\infty^2 - M^2\omega^2}\, \Phi_n(k_1, k_2, Mq_\infty, \omega), \qquad (41)$$

where $$\Phi_n = \Phi_0(k_1, k_2, Mq_\infty, \omega) = \frac{1}{2\pi}\int_0^\infty e^{-is} h_0(s)\, ds \quad \text{for } S0, \qquad (42)$$

$$\Phi_n = \Phi_1(k_1, k_2, Mq_\infty, \omega) = \frac{i}{\pi^{\frac{1}{2}}}\left\{\int_0^\infty A_i(-s)\, h_1(s)\, ds + \int_{-\infty}^0 A_i(s)\, h_1(s)\, ds\right\} \quad \text{for } S1, \qquad (43)$$

$$\Phi_n = \Phi_2(k_1, k_2, Mq_\infty, \omega) = \frac{1}{2\pi^{\frac{1}{2}}} e^{\frac{3}{8}\pi i} \int_{-1}^{1} \mathrm{Pa}\,(0) \frac{M^2 \Gamma(y, k_1, k_2, \omega)}{\psi'^{\frac{1}{2}} A^2}\, dy \quad \text{for } S2; \qquad (44)$$

$$k_\infty^2 = k_1^2 + k_2^2 + k_3^2.$$

In (41), the wave function is considered to be harmonic in the far field as well as in the near field. That is, the original wave function has been replaced by one which is harmonic throughout the entire space, while it matches the original solution in the far field.

By using (41), the spatial and temporal resolution of the solutions can be recovered by means of a four-dimensional inverse Fourier transformation:

$$\Phi_n(\mathbf{y}, t) = \iiiint \frac{(Mq_\infty)^{\frac{1}{2}}\, \Phi_n(k_1, k_2, Mq_\infty, \omega)}{k_\infty^2 - M^2\omega^2} \exp i(\mathbf{k}.\mathbf{y} + \omega t)\, d\mathbf{k}\, d\omega, \tag{45}$$

where the limits of integration are $\pm\infty$. This equation has exactly the same form, except for some numerical constants, as the general solution to a simple wave equation in three spatial dimensions (Morse & Feshbach 1953). This equation can then be written alternatively as

$$\begin{gathered} \Phi_n(\mathbf{y}, t) = \frac{1}{4\pi} \iiiint P(\mathbf{y}_0, \omega)\, G(\mathbf{y}, \mathbf{y}_0, \omega)\, e^{i\omega t}\, d\mathbf{y}_0\, d\omega, \\ P(\mathbf{y}_0, \omega) = \iiint \{Mq_\infty\}^{\frac{1}{2}}\, \Phi_n(\mathbf{k}, \omega)\, e^{+i\mathbf{k}.\mathbf{y}_0}\, d\mathbf{k}, \\ G(\mathbf{y}, \mathbf{y}_0, \omega) = e^{ik_\infty R}/R, \\ R = |\mathbf{y} - \mathbf{y}_0| \cong r - \mathbf{y}.\mathbf{y}_0/r \quad (\mathbf{r} = \mathbf{y};\, r = |\mathbf{r}|). \end{gathered} \tag{46}$$

In the above equations, the origin of the y co-ordinates is assumed to be in the source region. The dimension of the entire source region is assumed to be small compared with the distance from a point $\mathbf{y}_0$, in the source region, to a point $\mathbf{y}$ in the far field. Equation (45) can now be written as

$$\Phi(\mathbf{y}, t) = \frac{e^{ikr}}{4\pi r} \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} \{Mq_\infty\}^{\frac{1}{2}}\, \Phi_n(\mathbf{k}, \omega) \exp\left[-i\mathbf{y}_0 . \left(\mathbf{k} - \frac{k\mathbf{r}}{r}\right)\right] e^{i\omega t}\, d\mathbf{y}_0\, d\mathbf{k}\, d\omega. \tag{47}$$

If this equation is first integrated with respect to $\mathbf{y}_0$, and then integrated with respect to $\mathbf{k}$, the result can be written as

$$\Phi_n(\mathbf{y}, t) = \frac{e^{ikr}}{4\pi r} \int_{-\infty}^{\infty} \{Mq_\infty\}^{\frac{1}{2}}\, (2\pi)^3\, \Phi_n\left(\frac{k\mathbf{r}}{r}, \omega\right) e^{i\omega t}\, d\omega. \tag{48}$$

If noise spectrum is preferred, the above equation can be expressed as

$$\Phi_n(\mathbf{y}, \omega) = \frac{e^{ikr}}{4\pi r} (2\pi)^3 \{Mq_\infty\}^{\frac{1}{2}}\, \Phi_n\left(\frac{k\mathbf{r}}{r}, \omega\right). \tag{49}$$

Hence, the spatial resolution of the solutions to the convected wave equation has been recovered in terms of closed-form expressions.

Throughout the above analysis, it is not necessary to define the equivalent source function $P(\mathbf{y}_0, \omega)$. Therefore, any source function which can produce the correct spectral function in the far field will suffice.

Equation (49) indicates some important properties of the wave emission process. The intensity of the pressure wave depends on the inverse square of the radial distance between the source and the receiving point in the far field. Furthermore, the pressure fluctuation in the neighbourhood of any point in the far field is dominated by a thin pencil of rays emitted from the source in the direction of the receiving point. Since this solution is extremely similar in form to the solution written in the wavenumber co-ordinates, the inverse transformation has been overlooked in previous studies of the convected wave equation. After the inverse Fourier transformation, (48) and (49) show that the actual noise spectrum at a point differs from the plane wave spectrum, (15), (26) and (30), by a factor of k/r. As a consequence, some previous interpretations by Pao (1972) concerning the parametrical dependences of the solutions in various modes are in error.

Noise emission from random sources

For a random noise source, the source function can only be defined in a statistical sense. The results as obtained above have to be further qualified. It can be assumed that, for any of the above solutions, a complex conjugate problem of the wave propagation process can be posed. After the general solutions for both problems have been obtained, they can be multiplied together, and the ensemble average of the product can then be taken. The result will represent the statistical solution for noise emission from random sources. The $S0$ solution is chosen here to demonstrate the analysis. The general solution of the complex conjugate problem corresponding to the $S0$ case can be written as

$$\Phi_0^*(y, k_1, k_2, \omega) = \{Mq(y)\}^{-\frac{1}{2}}\left\{e^{-i\xi}\int_{-\infty}^{\xi} e^{+ir}\,h_0^*(r)\,dr + e^{i\xi}\int_{\xi}^{\infty} e^{-ir}\,h_0^*(r)\,dr\right\}. \quad (50)$$

Hence, the ensemble average of the product of these solutions can be written as

$$\overline{\Phi_0\Phi_0^*} = Mq(y)^{-1}\left\{\iint_{-\infty}^{\xi} e^{-is}\,e^{ir}\,\overline{h_0(s)\,h_0^*(r)}\,dr\,ds + \iint_{\xi}^{\infty} e^{is}\,e^{-ir}\,\overline{h_0(s)\,h_0^*(r)}\,dr\,ds\right\}, \quad (51)$$

where the ensemble average, denoted by an overbar, of the product of disjoint integrals is assumed to be zero. If the turbulence is assumed to be locally homogeneous in structure, the correlation coefficient of the turbulence at two separate points depends only on the distance of separation, and the intensity of turbulence depends only on the mean value of the co-ordinates of these points. A change of variable can be defined:

$$\lambda_3 = y^* - y, \quad \mu = \tfrac{1}{2}(y + y^*), \quad (52)$$

where y^* and y are points along the y axis which correspond to r and s, respectively, along the ξ co-ordinate.

The correlation function of the turbulence can be written in these co-ordinates as

$$\overline{h(s, k_1, k_2, \omega)\,h^*(r, k_1, k_2, \omega)} = N(\mu)\,\Pi(\lambda_3), \quad (53)$$

where $\Pi(\lambda_3)$ is the correlation function and $N(\mu)$ indicates the source strength distribution. For a small source volume, $N(\mu)$ can be defined as $N(\mu) = 1$ in the source region, and vanishes elsewhere. Unfortunately, (51) cannot be further simplified because the co-ordinate transformation between y and ξ is neither linear nor homogeneous. The origin of the ξ co-ordinate is always fixed at the transition point. A convolution integral for (51) is therefore impossible to derive except for special cases. For practical problems, the integrals will be evaluated by numerical calculations.

The higher iterations of the resolvent kernel

The resolvent kernel of a Volterra integral equation can be obtained from $K(\xi, s)$ by iteration:

$$R(\xi, s) = \sum_{n=1}^{\infty} K^n(\xi, s), \tag{54}$$

where $$K^n(\xi, s) = \int_s^\xi K(\xi, r)\, g(r)\, K^{n-1}(r, s)\, dr, \quad K^1(\xi, s) = K(\xi, s),$$

where $g(r)$ is the residue function of the WKBJ transformation as defined in (8). The subscript for this function is omitted here to avoid confusion. For the cases as studied in the present paper, the kernel of the integral equation can be written in general as

$$K(\xi, s) = F_2(\xi)\, F_1(s) - F_1(\xi)\, F_2(s), \tag{55}$$

where $F_1(\xi)$ and $F_2(\xi)$ are the normalized homogeneous solutions for each case. A formalism can now be established for the iterated kernels. If the functions are treated as components of a two-dimensional vector, then

$$K^n(\xi, s) = F_i(\xi)\, G_{ij}^n(\xi, s)\, F_j(s), \tag{56}$$

where $$G_{ij}^n(\xi, s) = \int_s^\xi G_{ik}^1(\xi, r)\, F_k(r)\, g(r)\, F_l(r)\, G_{lj}^{n-1}(r, s)\, dr$$

and $$\{G_{ij}^1(\xi, s)\} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

The summation convention is adopted for these equations. Since $g(\xi)$ approaches zero like ξ^{-2} for $\xi \to \infty$, the iteration is expected to converge rapidly.

There are two main reasons for writing the resolvent kernel in the present form. First, this formalism can be adapted directly for numerical computations. Second, an explicit form of the functional dependence of the resolvent kernel is necessary in order to show that the iteration will change only the source integral and not the form of the radiation boundary conditions in the far field.

The radiation boundary condition in the far field requires the knowledge of both $\Phi(\xi)$ and its first derivative. Using the nth approximation of the resolvent kernel, the solution to the wave equation can be written as

$$\Phi(\xi) = aF_1(\xi) + bF_2(\xi) + F_i(\xi) \int_0^\xi \sum_1^n G_{ij}^n(\xi, s)\, F_j(s)\, h(s)\, ds. \tag{57}$$

Hence, the derivative of $\Phi(\xi)$ depends on both the derivative of $F_i(\xi)$ and the derivative of the integral. The derivative of the integral contains two terms:

$$\frac{d}{d\xi} \int_0^\xi \sum_1^n G_{ij}^n(\xi, s)\, F_j(s)\, h(s)\, ds = \sum_1^n G_{ij}^n(\xi, \xi)\, F_j(\xi)\, h(\xi) + \int_0^\xi \sum_1^n \frac{d}{d\xi} G_{ij}^n(\xi, s)\, F_j(s)\, h(s)\, ds. \tag{58}$$

According to the definition of $G_{ij}^n(\xi, s)$, both $G_{ij}^n(\xi, s)$ and its first derivative vanishes at infinity. Hence, the derivative of the integral vanishes at infinity. The boundary conditions are defined by the homogeneous solutions and their derivatives. The analytical form of the radiation boundary conditions are, therefore, not affected by the order of iteration.

4. Discussion

Refraction effects

It has been pointed out earlier that the key wave propagation properties in a shear flow are fixed by the reduced wave equation. Some properties, such as the Doppler shift and the dynamical nature of source elements in a local frame of reference, have been discussed in connexion with $S0$. Closer examination of relations between the source function and far-field plane-wave elements will be given in this section.

In $S0$, the source element for noise radiation in the high frequency limit can be identified as an acoustic component of the local turbulence structure. Hence, such an element has also a local direction of propagation. If the local convection Mach number is M_c, then the local direction of propagation θ_0 will be related to the far-field angle of radiation θ by

$$\cos\theta_0 = A\cos\theta/(1-M_c\cos\theta). \tag{59}$$

This relation is obtained through (19) and the following formulae:

$$k_\infty\cos\theta = k_0\cos\theta_0, \quad k_\infty = M\omega, \quad k_0 = M\omega_0/A, \tag{60}$$

where k_0 is the magnitude of the wavenumber vector of the source element. It should be noted that (59) is identical to the refraction relation for wave propagation in a shear flow. Hence, the refracted path of wave propagation in the shear layer can be traced by means of (59). The above discussion applies also to the noise radiation mechanisms for source functions located in the hyperbolic branch of $S1$.

For sound sources in the elliptic branch of $S1$, the interpretation can be given from a different point of view. Analytically, the wavenumber component in the transverse direction is imaginary. It is well known that in such cases the wave particle velocity is out of phrase with the pressure by $\frac{1}{2}\pi$. There is no energy flux in this direction. However, the wavenumber component k_1 in the direction of the flow is real, and the turbulent energy contained within a given source element will be carried forwards through local pressure fluctuations. For a small source volume located beneath the transition point, its pressure fluctuation is felt, with an exponential decay factor, by the fluid layer in the neighbourhood of the transition point. At the transition point, the pressure fluctuations attain the sonic phase velocity. Hence, the pressure fluctuations can now propagate and turn as an acoustic wave, and leave the shear layer as a radiated plane-wave element. For source elements which are located in the elliptic branch of $S1$, equation (59) fails because $|\cos\theta_0| > 1$. Here k_0 is no longer related to ω_0 through the local speed of sound as specified by (60).

For source volumes in $S2$, the transverse wavenumber component cannot be precisely defined. Nevertheless, it can be considered as zero. The turbulence source fluctuation has a real propagation velocity in the direction of the flow. The general solution in $S2$ indicates that the source function in such cases is coupled directly with radiated plane-wave elements in the far field.

The correspondence between the source and the radiated noise as described here is not the same as that given in the classical analysis of turbulent noise. In both theories, the frequencies are related through the Doppler shift relation. In the classical analysis, the wavenumber vector is the same for a plane-wave element in the far field as for the corresponding turbulent source component. The analysis which leads to this conclusion is similar to the derivation of (48) in connexion with the inverse Fourier transformation. This is perhaps a necessary consequence if the wave propagation process is governed by the simple wave equation throughout the entire space. Owing to such correspondences between wavenumber and frequency, the phase speed for the source component in the convected frame of reference can range from zero to values much greater than the local speed of sound. This phase speed equals the local speed of sound only if $\theta = \frac{1}{2}\pi$, and it is greater than the ambient speed of sound for all $\theta > \frac{1}{2}\pi$. In the present theory, the local phase speed of the source element can only be equal to or smaller than the local speed of sound. In the case of radiation in upstream directions, the transverse wavenumber component changes rapidly in the shear layer such that locally the ratio of frequency to the magnitude of the wavenumber vector is always the speed of sound. The physical interpretation of such requirements appears to be correct. The turbulent source is embedded entirely in the fluid and is not in contact with any solid surface. In addition, only linear wave propagation is considered. Hence, the fluctuations in the turbulence can only have phase speeds which are equal to or less than the local speed of sound.

So far, noise radiation has been considered in one direction at a time. The solutions are given in forms such that all contributions from various source volumes are summed along the path of radiation which leads to a given direction in the far field. On the other hand, it is more familiar to consider a given compact source volume which radiates noise in all directions. It is clear from the above discussions that different spectral components of this volume of turbulence will be responsible for noise radiation in various frequencies. The noise radiation mechanism will also be governed by different modes of solution to the convected wave equation.

For a given convection Mach number for the source volume, the far field can be divided into a maximum of four zones. The governing type of solution which relates the radiated noise to the source function will be different in each zone. The dividing angles θ_1, θ_2 and θ_3 for these zones are given by

$$\left.\begin{aligned} \cos\theta_1 &= 1/(M_c - A), \quad \theta_1 = 0 \quad \text{if} \quad (M_c - A) \leqslant 1,\\ \cos\theta_2 &= 1/(M_c + A),\\ \cos\theta_3 &= 1/(M_c - A), \quad \theta_3 = \pi \quad \text{if} \quad (M_c - A) \geqslant -1.\end{aligned}\right\} \tag{61}$$

For sound radiation between $\theta = 0$ and θ_1 the source element is locally acoustic. However, the pressure wave has to pass through a pair of transition points in order to reach the far field. For high frequency radiation, the pressure wave will be heavily attenuated as it passes through the elliptic segment between the transition points. For low frequency radiation, the radiation mechanism will be governed by $S2$. This zone exists only if the convection Mach number is sufficiently large such that $M_c - A$ is greater than one. In the second zone bounded between θ_1 and θ_2, the source element is hydrodynamic. Here the noise radiation process is governed by either $S1$ or $S2$, depending on the frequency. It should be noted that θ_2 is always smaller than $\frac{1}{2}\pi$. For noise emission between θ_2 and θ_3, the source element is locally acoustic. Since there is no transition point between the source and the far field, the sound radiation process is governed by $S0$. A fourth zone exists between θ_3 and $\theta = \pi$ in the special case where the flow temperature is high while the convection Mach number is small such that $M_c - A$ is less than -1. The source element for sound emission in this zone is again hydrodynamic, and wave propagation will be governed by $S1$ or $S2$. In most cases, only θ_2 has a non-trivial value. The noise radiation mechanisms will be only those described above for the second and third zones.

The boundary between zones should not be considered as a sharp line of separation. For example, the source components in the third zone which are responsible for noise radiation near θ_2 and θ_3 are indeed acoustic in nature. However, the source volume is located very close to a transition point. It is more appropriate to describe the radiation process by means of $S1$.

Convection effects

In discussions below, the value of k_2 will be taken as zero. This assumption is made mainly for convenience. With this restriction, the paths of noise radiation will fall on a plane which also contains the normal and the y_1 axis of the shear layer. If the source function is confined to a small volume near the origin, a rotation of co-ordinates in the plane of the shear layer will make any radiation path coincide with such a vertical plane. The value of k_2 in the new co-ordinates will be zero. In this case, the convection Mach number M_c will be replaced by its effective component $M_c \cos\phi$, where ϕ is the angle of rotation.

A simple turbulence structure can now be introduced to serve as the source function:

$$\overline{v_i v_j}(\boldsymbol{\lambda}, \tau) = v_0^2 \left\{\left(1 - \frac{\lambda^2}{L_1^2}\right)\delta_{ij} + L_1^{-2}\lambda_i\lambda_j\right\} \exp\left[-\left\{\left(\frac{\lambda}{L_1}\right)^2 + \left(\frac{\tau}{L_t}\right)^2\right\}\right], \tag{62}$$

or

$$\overline{v_i v_j}(\mathbf{k}, \omega) = \frac{v_0^2 L_1^3 L_t}{32\pi^2}\{(kL_1)^2\delta_{ij} - L_1^2 k_i k_j\} \exp\left[-\tfrac{1}{4}\{(kL_1)^2 + (\omega L_t)^2\}\right], \tag{63}$$

where $\boldsymbol{\lambda}$ is the spatial separation from the source, τ is the time delay, L_1 is the spatial scale of the turbulence and L_t is the time scale of the turbulence. This turbulence structure satisfies only the equation of continuity of an incompressible fluid. The quadrupole self-noise source can now be written in more precise terms:†

$$\overline{\frac{\partial v_i}{\partial y_j}\frac{\partial v_j}{\partial y_i}\frac{\partial v_k^*}{\partial y_l^*}\frac{\partial v_l^*}{\partial y_k^*}} = \frac{\partial^4}{\partial\lambda_i\partial\lambda_j\partial\lambda_k\partial\lambda_l}\overline{v_i v_j v_k^* v_l^*}(\boldsymbol{\lambda}, \tau), \tag{64}$$

where the turbulence is assumed to be locally incompressible. According to Batchelor (1960, p. 179) the fourth-order correlation can be related to the second-order correlations via

$$\overline{v_i v_j v_k^* v_l^*} = \overline{v_i v_j} \,.\, \overline{v_k^* v_l^*} + \overline{v_i v_k^*} \,.\, \overline{v_j v_l^*} + \overline{v_i v_l^*} \,.\, \overline{v_j v_k^*}. \tag{65}$$

† The shear noise term will not be presented in this paper. Its analytical form is much simpler, while its functional dependence is similar to that of the self-noise term. See Pao & Lowson (1970).

If $k_2 = 0$, then only the derivatives with respect to y_1 and y_3 need to be considered. Equation (64) represents a contraction of the fourth-order correlation function Ψ which contains sixteen terms. These terms can be evaluated through convolution integrals, and their sum turns out to be rather simple:

$$\Psi(\lambda_3, k_1, k_2, \omega) = \frac{v_0^4 L_t}{16\pi(2\pi)^{\frac{1}{2}} L_1^2}\left\{(k_1 L_1)^4 - 2(k_1 L_1)^2 \frac{\partial^2}{\partial\lambda_3^2} + \frac{\partial^4}{\partial\lambda_3^4}\right\} \times \exp\left[-2\left(\frac{\lambda_3}{L_1}\right)^2 - \tfrac{1}{8}\{(k_1 L_1)^2 + (\omega_0 L_t)^2\}\right]. \quad (66)$$

It should be noted that k_2 has been considered as a variable in the convolution integrals. It is given the value of zero only after all the integrations have been completed.

Equation (66) represents the ensemble average of the square of the self-noise term which appears in the convected wave equation. Therefore, it can be used directly in the calculation of the mean-square value of the sound pressure. A formula for the mean-square value of the pressure field has been derived for $S0$ in the previous section. By using the same procedure, it is equally easy to derive formulae for the spectral density function $\overline{\Phi_n \Phi_n^*}(\mathbf{y}, \omega)$ which describes the noise spectrum received at a point in the far field. The general formula for $\Phi_n(\mathbf{y}, \omega)$ is given by (49), and the source integrals for $S0$, $S1$ and $S2$ are given by (42)–(44). A symbolic expression for the mean-square value of the source integral can be written as

$$\overline{\Phi_n \Phi_n^*}(\mathbf{k}, \omega) = \iint F_1(r) F_1^*(s) \frac{M^4 \Psi(y^* - y, k_1, k_2, \omega)}{\{\psi'(y)\,\psi'(y^*)\}^{\frac{1}{2}} A^2}\, dy\, dy^*, \quad (67)$$

where $F_1(\xi)$ stands for $\exp\{\pm i\xi\}$, $\mathrm{Ai}(\pm\xi)$, $\mathrm{Pa}(\xi)$ or $\mathrm{Qa}(\xi)$, whichever is appropriate; $F_1^*(\xi)$ is the complex conjugate of $F_1(\xi)$; and the variables r and s are dummies corresponding to the transformed co-ordinate ξ. There are two cases where this integral can be estimated analytically. In the first case, the source function is confined to the neighbourhood of a transition point. In the second the source volume is small and located sufficiently far from a transition point, and it is required also that the function $q(y)$ be approximately constant throughout the source region. The functional dependence of the above integral can reveal important information concerning the convection laws for each of the three solutions. Hence, the estimate of this integral will be examined separately for each case.

If in $S0$ the source volume is sufficiently small, the value of Mq will appear to be constant across the source volume. Consequently, the variables r and s will be related linearly to y and y^*. The function $F_1(r)\,F_1^*(s)$ can be written as a function of λ_3 and μ, which are defined in (52). After a simple analysis, it can be shown that the integral is equivalent to a local Fourier transformation in the variable λ_3. Furthermore, $\psi' = Mq_0$ in the source region. Hence,

$$\begin{aligned}\overline{\Phi_0 \Phi_0^*}(\mathbf{y}, \omega) &= (16\pi^2 r^2)^{-1} (Mq_\infty)\,(2\pi)^6\, \overline{\Phi_0 \Phi_0^*}(\mathbf{k}, \omega) \\ &= \frac{(2\pi)^6}{16\pi^2 r^2}\left\{\frac{Mq_\infty}{Mq_0}\right\} \frac{v_0^4 M^4 L_t}{128\pi^3 L_1 A^2}\{(k_1 L_1)^4 + 2(k_1 L_1)^2 (Mq_0 L_1)^2 + (Mq_0 L_1)^4\} \\ &\qquad \times \exp\left[-\tfrac{1}{8}\{(k_0 L_1)^2 + (\omega_0 L_t)^2\}\right] \\ &= \frac{\pi v_0^4 M^4 L_t}{32 r^2 L_1 A^2}\left(\frac{q_\infty}{q_0}\right)(k_0 L_1)^4 \left[\exp - \tfrac{1}{8}\{(k_0 L_1)^2 + (\omega_0 L_t)^2\}\right]. \end{aligned} \quad (68)$$

By comparing this equation with the classical results, one finds that the convection law is tremendously different for the present equation. In the classical result, the factor $(k_0 L_1)^4$ will be replaced by $(k_\infty L_1)^4$; the term $(k_\infty L_1)^2$ in the exponential index will be replaced by $(k_\infty L_1)^2$, so that

$$\overline{\Phi_0 \Phi_0^*}(\mathrm{y}, \omega) \sim M^8(1 - M_c \cos\theta)^{-5} (\omega_0 L_1)^4 \exp[-\tfrac{1}{8}\{(k_\infty L_1)^2 + (\omega_0 L_t)^2\}], \quad (69)$$

where a bandwidth adjustment of $(1 - M_c \cos\theta)^{-1}$ has been included. If (68) is written in the same form, the convection factor of $(1 - M_c \cos\theta)^{-4}$ will be absent from the expression for sound intensity:

$$\overline{\Phi_0 \Phi_0^*}(\mathrm{y}, \omega) = \frac{\pi v_0^4 M^8 L_t}{32 r^2 A^6 L_1} \left(\frac{q_\infty}{q_0}\right) \frac{(\omega_0 L_1)^4}{(1 - M_c \cos\theta)} \exp[-\tfrac{1}{8}\{(k_0 L_1)^2 + (\omega_0 L_t)^2\}]. \quad (70)$$

In (68), there is a factor of q_∞/q_0. It is natural to ask what may happen if q_0 vanishes. If this is the case, the source volume will be in the immediate neighbourhood of a transition point, the $S0$ solution will not be valid. Radiation from such source volumes should be calculated by using the $S1$ or $S2$ solution. In the above analysis, the discussion should be valid for all frequencies as long as the source volume remains compact and small. That is, it is not necessary to confine the analysis to the high frequency limit. However, this analysis is not valid for very low frequencies where the WKBJ transformation itself would fail.

In the $S2$ case, the values of r and s are small throughout the shear layer. Hence, the function $F_1(r) F_1^*(s)$ can be replaced by its value at $r = s = 0$. The integral in (67) can be written simply as an integration of the source function over the variables λ_3 and μ. For convenience, the source volume is assumed to be small such that the convection Mach number will be approximately constant for the entire source volume. While this assumption allows a straightforward interpretation of the result, it is not essential to the analysis. The value of ψ' is proportional to $k_1^{\frac{1}{2}}(M\Omega)^{\frac{1}{2}}$ in the present case:†

$$\psi' = \kappa\{M k_1 \Omega\}^{\frac{1}{2}}, \quad \Omega = \frac{1}{M} \frac{d}{dy}\left(\frac{M_c}{A}\right), \quad (71)$$

where κ is a proportionality constant which equals one in the neighbourhood of the transition point. It should be noted that the integral in (67) in its present form can be considered as a Fourier transformation in the λ_3 co-ordinate, while the wavenumber component in this direction vanishes. The result of the integration can now be written as

$$\begin{aligned}\overline{\Phi_2 \Phi_2^*}(\mathrm{y}, \omega) &= \frac{(2\pi)^6 M q_\infty}{16\pi^2 r^2} \overline{\Phi_2 \Phi_2^*}\left(\frac{k\mathbf{r}}{r}, \omega\right) \\ &= \frac{\pi^2 M^4 v_0^4 L_t}{32 r^2 A^2 L_1} \left(\frac{M q_\infty}{\psi'}\right) \mathrm{Pa}^2(0)\, (k_1 L_1)^4 \exp[-\tfrac{1}{8}\{(k_1 L_1)^2 + (\omega_0 L_t)^2\}] \\ &= \frac{\pi^2 M^{\frac{7}{2}} v_0^4 L_t}{32 r^2 A^2 L_1} \left\{\frac{\tan\theta}{L_1^{\frac{1}{2}} \Omega^{\frac{1}{2}}}\right\} \mathrm{Pa}^2(0)\, (k_1 L_1)^{\frac{9}{2}} \exp[-\tfrac{1}{8}\{(k_1 L_1)^2 + (\omega_0 L_t)^2\}].\end{aligned} \quad (72)$$

Again, the factor $M q_\infty / \psi'$ deserves some attention. It remains finite for all directions of wave radiation which belong to $S2$. The domain of validity for $S2$ has

† There are two cases of $S2$. In the other one, $\psi' = k^{\frac{1}{3}}\{\frac{1}{2} M \Omega'\}^{\frac{1}{4}}$.

been defined in the last subsection. Here θ will always be smaller than $\frac{1}{2}\pi$, and it can approach $\frac{1}{2}\pi$ only if the value of M_c approaches infinity.

If the wavenumber factor is rewritten in terms of the local frequency ω_0, equation (72) will read

$$\overline{\Phi_2 \Phi_2^*}(\mathbf{y}, \omega) = \frac{\pi^2 M^8 v_0^4 L_t \,\mathrm{Pa}^2(0)}{32 r^2 A^2 L_1^{\frac{3}{2}} \Omega^{\frac{1}{2}}} \frac{(\omega_0 L_1 \cos\theta)^{\frac{9}{2}}}{\{1 - M_c \cos\theta\}^{\frac{11}{2}}} \exp\left[-\tfrac{1}{8}\{(k_1 L_1)^2 + (\omega_0 L_t)^2\}\right]. \qquad (73)$$

The convection law as shown in this equation is different from the classical result in two ways. First, there is an additional factor of $(\cos\theta)^{4\cdot5}$. Second, the convection factor is stronger by a factor of $(1 - M_c \cos\theta)^{-\frac{1}{2}}$. Analytically, the latter effect is a result of the interaction between the noise radiation mechanism and the shear velocity gradient, since an additional factor of $(k_1 L_1)^{\frac{1}{2}}$ comes from the Mq_∞/ψ' factor in (72).

The $S1$ case will be considered in two parts. Again, the source volume is assumed to be small: it is located either in the neighbourhood of the transition point or far away from it. The convection effects associated with the former condition can be analysed in the same way as is the $S2$ case. Here, the value of ψ' in the neighbourhood of the transition point is

$$\psi' = k^{\frac{2}{3}} (2M\Omega)^{\frac{1}{3}} \qquad (74)$$

and the function $F_1(r)\, F_1^*(s)$ will be evaluated at $r = s = 0$. Hence, (67) can be written as

$$\begin{aligned} \overline{\Phi_1 \Phi_1^*}(\mathbf{y}, \omega) &= \frac{\pi^2 v_0^4 M^{\frac{11}{3}} L_t \mathrm{Ai}^2(0) \tan\theta}{8 r^2 A^2 L_1^{\frac{4}{3}} \Omega^{\frac{1}{3}}} (k_1 L_1)^{\frac{13}{3}} \exp\left[-\tfrac{1}{8}\{(k_1 L_1)^2 + (\omega_0 L_t)^2\}\right] \\ &= \frac{\pi^2 v_0^4 M^8 L_t \mathrm{Ai}^2(0) \tan\theta}{8 r^2 A^2 L_1^{\frac{4}{3}} \Omega^{\frac{1}{3}}} \frac{(\omega_0 L_1 \cos\theta)^{\frac{13}{3}}}{(1 - M_c \cos\theta)^{\frac{16}{3}}} \exp\left[-\tfrac{1}{8}\{(k_1 L_1)^2 + (\omega_0 L_t)^2\}\right]. \end{aligned} \qquad (75)$$

In (75) the convection factor is different from the classical result by a factor of $(\cos\theta)^{4\cdot33}(1 - M_c \cos\theta)^{-0\cdot33}$. The interaction between the sound emission process and the velocity gradient is slightly weaker than in the $S2$ case.

If the source volume is located in the elliptic branch and far away from the transition point, the Airy functions can be replaced by their asymptotic forms. Consequently,

$$\begin{aligned} F_1(r)\, F_1^*(s)/\{\psi'(r)\, \psi'(s)\}^{\frac{1}{2}} &= r^{\frac{1}{4}} s^{\frac{1}{4}} F_1(r)\, F_1^*(s)/\{M^2 q(r)\, q(s)\}^{\frac{1}{2}} \\ &= \frac{1}{4\pi M q_0} \exp\left\{-\int^{y} Mq\, dy\right\} \exp\left\{-\int^{y^*} Mq\, dy\right\}. \end{aligned} \qquad (76)$$

Since the source volume is small and q remains approximately constant over the source volume, the argument of the exponential function can be written as

$$\int^{y} Mq\, dy = \int^{y_1} Mq\, dy + \int_{y_1}^{y} Mq\, dy = Q + Mq_0(y - y_1),$$

$$Q = \int^{y_1} Mq\, dy, \qquad (77)$$

where y_1 is a suitable point in the source volume and q_0 is the value of q at y_1.

It is important to note that the above function depends only on μ and not λ_3.

The integration of (67) over the variables λ_3 and μ has again the meaning of a Fourier transformation of the source function in the λ_3 co-ordinate, while the wavenumber component vanishes. The result of the integration can now be given as

$$\overline{\Phi_1 \Phi_1^*}(\mathbf{y}, \omega) = \frac{\pi v_0^4 M^8 L_t q_\infty}{32 r^2 A^2 L_1 q_0} e^{-2Q} \frac{(\omega_0 L_1 \cos\theta)^4}{(1-M_c\cos\theta)^5} \exp[-\tfrac{1}{8}\{(k_1 L_1)^2 + (\omega_0 L_t)^2\}]. \quad (78)$$

The convection factor $(1-M_c\cos\theta)^{-5}$ is exactly the same as in the classical results. However, the above equation contains also a factor of $(\cos\theta)^4 e^{-2Q}$. From the definition of transition point, one can verify that the transition point is closer to the edge of the shear layer for smaller values of θ. If the source volume is located at a fixed depth in the shear layer, the distance between the source and the transition point will be greater for smaller angles of radiation. For a given frequency, the directivity of noise radiation will be strongly modified by the factor e^{-2Q}. This is perhaps a contributing factor which leads to the observed cardioid pattern of jet noise radiation at high frequencies.

Q is a linear function of ω_0:

$$Q = k_1 W(y_0, \theta) = \frac{M\omega_0 \cos\theta}{(1-M_c\cos\theta)} W(y_0, \theta), \quad (79)$$

where y_0 is the position of the source function and θ is the angle of noise radiation in the far field. By means of this definition, (78) can be integrated over ω_0 in closed analytical form.

Mach-number and temperature dependence

Equations (70), (73), (75) and (78) can be integrated analytically over ω_0. The result will provide a value for the total amount of noise emission in a given direction for each of these cases. The exponential function in the turbulent source spectrum can be written as a function which depends only on ω_0:

$$\exp[-\tfrac{1}{8}\{(k_0 L_1)^2 + (\omega_0 L_t)^2\}] = \exp\left[-\frac{1}{8}\left(\frac{M}{M_c}\right)^2 \left(\frac{\omega_0 L_1}{\alpha A}\right)^2 (\alpha^2 M_c^2 + A^2)\right] \quad \text{for } S0, \quad (80)$$

$$\exp[-\tfrac{1}{8}\{(k_1 L_1)^2 + (\omega_0 L_t)^2\}] = \exp\left[-\frac{1}{8}\left(\frac{M}{M_c}\right)^2 \left(\frac{\omega_0 L_1}{\alpha}\right)^2 \times \frac{(1-M_c\cos\theta)^2 + \alpha^2 M_c^2 \cos^2\theta}{(1-M_c\cos\theta)^2}\right] \quad \text{for } S1 \text{ and } S2, \quad (81)$$

$$L_1/L_t = \alpha(M_c/M),$$

where α is a constant which defines the ratio of the integral spatial scale and the integral time scale of the given turbulence structure. The value of α^2 can be taken as 0·1 in jet turbulence. The value of M/M_c is a matter of definition. If a single source volume is under consideration, the reference Mach number M can be defined as the same as the local convection Mach number M_c. Hence, $M/M_c = 1$. Equations (70), (73), (75) and (78) can now be integrated for noise emission in a given direction. The results are, respectively,

$$\overline{\Phi_0 \Phi_0^*}(\mathbf{y}) = \frac{3 \times 2^{\frac{1}{2}} \pi^{\frac{3}{2}} v_0^4 M^8 \alpha^4 q_\infty}{r^2 A^2 L_1 q_0} \left(\frac{M_c}{M}\right)^4 \frac{A}{(1-M_c\cos\theta)} \{A^2 + \alpha^2 M_c^2\}^{-\frac{5}{2}} \quad \text{for } S0, \quad (82)$$

$$\overline{\Phi_2\,\Phi_2^*}(y) = \frac{21\times 2^{\frac{1}{2}}\pi^2 v_0^4 M^8\,\mathrm{Pa}^2(0)\,\alpha^{\frac{9}{2}}\tan\theta}{4r^2A^2L_1^{\frac{3}{2}}\Omega^{\frac{1}{2}}}\Gamma(\tfrac{3}{4})\{\cos\theta\}^{\frac{9}{2}}\left(\frac{M_c}{M}\right)^{\frac{9}{2}}$$
$$\times\{(1-M_c\cos\theta)^2+\alpha^2M_c^2\cos^2\theta\}^{-\frac{11}{4}}\quad\text{for } S2, \tag{83}$$

$$\overline{\Phi_1\,\Phi_1^*}(y) = \frac{160\times 2^{\frac{1}{2}}\pi^2 v_0^4 M^8\,\mathrm{Ai}^2(0)\,\alpha^{\frac{13}{3}}\tan\theta}{9r^2A^2L^{\frac{4}{3}}\Omega^{\frac{1}{3}}}\Gamma(\tfrac{2}{3})\{\cos\theta\}^{\frac{13}{3}}\left(\frac{M_c}{M}\right)^{\frac{13}{3}}$$
$$\times\{(1-M_c\cos\theta)^2+\alpha^2M_c^2\cos^2\theta\}^{-\frac{8}{3}}\quad\text{for } S1\,(a), \tag{84}$$

$$\overline{\Phi_1\,\Phi_1^*}(y) = \frac{3\times 2^{\frac{1}{2}}\pi^{\frac{3}{2}}v_0^4 M^8\,\alpha^4 q_\infty}{r^2A^2L_1q_0}\left(\frac{M_c}{M}\right)^4 F(b)\cos^4\theta$$
$$\times\{(1-M_c\cos\theta)^2+\alpha^2M_c^2\cos^2\theta\}^{-\frac{5}{2}}\quad\text{for } S1\,(b), \tag{85}$$

where $\Gamma(x)$ denotes the gamma function, and the function $F(b)$ and the value of b are defined as

$$F(b) = \int_b^\infty e^{b^2-u^2}(u-b)^4\,du, \tag{86}$$

$$b = \frac{2^{\frac{3}{2}}\,M\,W(y_0,\theta)\cos\theta}{L_t\{(1-M_c\cos\theta)^2+\alpha^2M_c^2\cos^2\theta\}^{\frac{1}{2}}}. \tag{87}$$

It should be noted that the value of b in (87) is defined for self-noise. Its value is smaller for shear noise by a factor of $1/2^{\frac{1}{2}}$. In (82), there is a factor of $(1-M_c\cos\theta)^{-1}$. It is important to note that this term remains finite for all solutions which belong to $S0$. According to (59) and (61), the value of $(1-M_c\cos\theta)^{-1}$ can never be greater than $(M_c+A)/A$. For small values of Mach number, all of the above cases obey an M^8 law. For high Mach numbers, the current results are not the same as the classical results. First of all, the $S0$ case approaches an M^3 law only when M_c^2 becomes significantly greater than A^2. However, this does not mean that $S0$ radiation is necessarily more efficient in the higher Mach number range than $S1$ or $S2$, because this type of noise radiation did not have the advantages of a convection factor to begin with as did the other two cases. It is quite surprising to note that the convection law for the $S2$ and $S1\,(a)$ case turns out to be less efficient than the M^3 law in the high Mach number range: there is an $M^{2{\cdot}5}$ law for $S2$ and an $M^{2{\cdot}67}$ law for $S1\,(a)$. For the $S1\,(b)$ case, the Mach-number dependence is identical to the classical results.

It is not clear whether the slight decline of the noise radiation efficiency in the $S1\,(a)$ and $S2$ cases is experimentally observable, assuming that the above conclusion is correct. The overall noise radiation depends also on the fourth power of the turbulence intensity v_0. If this non-dimensional value depends on even a small positive fractional power of the Mach number, the dependence of v_0^4 on Mach number will largely compensate for small deviations from the M^3 law.

Another unexpected result in this study is the dependence of the radiated sound pressure on c_j/c_0. In (1), both the source function $\gamma(\partial v_i/\partial y_j)\,\partial v_j/\partial y_i$ and the pressure fluctuation $\log(p/p_0)$ are multiplied by a factor A. Since the source function as defined in (8) depends explicitly on A^{-2}, it may appear that the intensity of the radiated noise would depend on A^{-4}. This is not the case if one examines carefully the solutions as given by (15), (26), (36) and (39). While the factor AA^{-2} of the source integrand is evaluated in the shear flow region, the factor A associated with $\log(p/p_0)$ should be evaluated in the far field, where it has a value of one. Hence, the sound intensity in the far field depends only on A^{-2} instead of A^{-4}. Such a dependence can be written as either T/T_j or ρ_j/ρ_0, and it is much weaker than the corresponding dependence discussed in Ribner (1964).

Limitations to the present theory

In general, the accuracy of the present analysis is limited to the extent where (3) is applicable. Although there may be further improvements of this wave equation under special flow conditions, it appears that the present solution contains a sufficient number of differences from the classical analysis to warrant detailed considerations.

The second limitation of the present analysis is inherent in the WKBJ transformation. The application of the WKBJ method should be restricted to cases where $kL > 1$, where k denotes here a dimensional wavenumber. The value of kL is directly related to the Strouhal number St in aerodynamic noise:

$$St = \omega L/2\pi U = kL/2\pi M \quad (kL = 2\pi St M).$$

For a turbulent round jet, the characteristic dimension is the exit diameter of the jet. For most jets, the peak Strouhal number is between 0·25 and 0·30. If the jet velocity is greater than 0·6 times the ambient speed of sound, all the noise radiation near or above the peak of the spectrum can be studied by the present theory. The same statement can be made also for rocket noise radiation. The peak Strouhal number for rocket noise ranges from 0·04 down to 0·02. However, the corresponding values of M ranges from a minimum of 6 to well above 10.

In the final discussion, most of the interpretations are made on the assumption that the source volume is small. If the source volume is large, much of the simple interpretation in terms of ray acoustics will no longer apply. Hence, such discussion serves only to indicate the underlying mechanisms of noise emission from a physical point of view of fluid mechanics.

Finally, it should be pointed out that the present formulation is given in terms of Cartesian co-ordinates. It is perhaps more appropriate to study problems such as jet noise radiation in terms of cylindrical co-ordinates. Mechanisms such as low frequency noise radiation and the interaction between jet instability and noise radiation can be described more accurately in cylindrical co-ordinates. Analytical techniques such as Fourier transformations and WKBJ methods remain available for obtaining solutions from the latter case.

This work was supported by the Unsteady Gasdynamics Branch of National Aeronautics and Space Administration–Marshall Space Flight Center, under contract no. NAS 8-28588.

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